Lecture Notes on Topological Insulators

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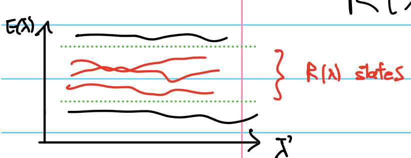
Reference Material

* Bradlyn, Iraola ; SciPost Lect. Notes 51 (2022)
* Sergeev ; SciPost Lect. Notes 67 (2023)
* Asbóth, Oroszlány, Pályi, Lect. Notes in Physic (2016) Springer → arXiv 1509.02295
* Bernevig, Hughes ; Topological Insulators and Superconductors, Princeton (2013)

Lecture 1 - Adiabatic Transport & Geometric Phase

1 - Ingredients

* A smooth manifold $\mathcal{M}$, where parameters $\vec{\lambda}$ live
* Hamiltonian family dependent on $\vec{\lambda}$: $H(\vec{\lambda})$
* At every $\vec{\lambda}$, $H(\vec{\lambda})$ has a discrete spectrum.
* Vector space spanned by $S(\vec{\lambda}) = \{ |\phi_i(\vec{\lambda})\rangle, \; \overset{i \in I}{i \in 1, \cdots, M} \}$
* Want to focus on a subspace $R(\vec{\lambda}) \supset S(\vec{\lambda})$, such that the bands are well separated from $\overline{R}(\vec{\lambda})$ $\forall \vec{\lambda} \in \mathcal{M}$: $R(\vec{\lambda}) = \{ |\phi_n(\vec{\lambda})\rangle, \; n = 1, \cdots, N < M \}$, $n \in Y$



$$\min_n |E(\vec{\lambda}) - E_n(\vec{\lambda})| \geq \Delta$$

where
$$H|\psi(\vec{\lambda})\rangle = E(\vec{\lambda})|\psi(\vec{\lambda})\rangle \quad \& \quad |\psi(\vec{\lambda})\rangle \in \overline{R}(\vec{\lambda})$$
$$H|\phi_n(\vec{\lambda})\rangle = E_n(\vec{\lambda})|\phi_n(\vec{\lambda})\rangle$$

* We can then work on this subspace by defining the projection operator

$$P = \oint_{C_{\vec{\lambda}}} dz \frac{1}{z - H(\vec{\lambda})}$$

$$= \sum_{n \in Y} |\phi_n(\vec{\lambda})\rangle\langle\phi_n(\vec{\lambda})|$$

$\longrightarrow$ Projectors allow for a simple geometric formulation of the adiabatic theorem.

2- Adiabatic Theorem

A physical system remains in its instantaneous eigenstate if a given perturbation is acting on it slowly enough and if there is a gap between the eigenvalue and the rest of the Hamiltonian's spectrum

→ Simple words: if system begins close to an eigenstate it remains close to an eigenstate.

* Quantitative statement:

$$\exists\, U_a(t) \mid \quad P(t) = U(t)\, P(0)\, U^\dagger(t)$$

$$\approx U_a(t)\, P(0)\, U_a(t) = P(\vec{\lambda}(t))$$

$$U_a(t)\, U^\dagger(t) - \mathbb{1} = \mathcal{O}(1/T)$$

where T is the total time of adiabatic evolution

* One says that $T \to \infty$ is the adiabatic limit

Proof: See lecture notes from Prof Wiles, Winter 2018, university Waterloo.

* Next: we want to find an expression for $U_a(t)$

3 - Geometric Evolution

$$* \quad \dot{P} = \dot{U}_a P_0 U_a^\dagger + U_a P_0 \dot{U}_a^\dagger$$

$$= \dot{U}_a U_a^\dagger U_a P_0 U_a^\dagger + U_a P_0 U_a^\dagger U_a \dot{U}_a^\dagger$$

$\partial_t (U_a U_a^\dagger) = 0 = \dot{U}_a U^\dagger + U_a \dot{U}_a^\dagger$

$$= \dot{U}_a U_a^\dagger U_a P_0 U_a^\dagger - U_a P_0 U_a^\dagger \dot{U}_a U_a^\dagger$$

$$= [\dot{U}_a U_a^\dagger , U_a P_0 U_a^\dagger] = [\dot{U}_a U_a^\dagger , P] \qquad (1)$$

* (1) is satisfied by $\dot{U}_a U_a^\dagger = [\dot{P}, P]$

(Verify! Hint: Show & use $P\dot{P}P = 0$)

* Diff eq: $\quad \dot{U}_a = [\dot{P}, P] U_a \equiv \mathcal{A}_s U_a$

$$\longrightarrow \quad U_a = \mathcal{T} \exp\left(\int_0^t dt' \, \mathcal{A}_s(t') \right)$$

* Using $\dfrac{dP}{dt} = \vec{\nabla} P \cdot \dfrac{d\vec{\lambda}}{dt}$, we can also write

$$\dot{U}_a = [\vec{\nabla} P, P] \cdot \dot{\vec{\lambda}} \, U_a$$

and $$U_a = \mathcal{P} \exp\left(\int_\gamma \vec{A}_s(\vec{\lambda}) \cdot d\vec{\lambda} \right)$$

where $\gamma : \{ \vec{\lambda}(t) , \; t' \in [0,T] \}$

Purely geometric quantity

4 - Geometric Action on States

* Question : How does the state behave as it moves through the manifold ?

$$\vec{\nabla} |\phi(\vec{\lambda})\rangle = \vec{\nabla} U_A |\phi_0\rangle \quad (1)$$

$$= \vec{A}_s U_A |\phi_0\rangle$$

$$= [\vec{\nabla} P, P] |\phi(\vec{\lambda})\rangle$$

* Using $P\dot{P}P = 0$,

$$[\vec{\nabla} P, P] |\phi(\vec{\lambda})\rangle = [\vec{\nabla} P, P] P |\phi(\vec{\lambda})\rangle \quad (2)$$

$$= (\vec{\nabla} P) |\phi(\vec{\lambda})\rangle \quad (3)$$

* (1) - (2) : $[\vec{\nabla} - \vec{A}_s] |\phi(\vec{\lambda})\rangle = 0$
* (3) - (2) : $P \vec{\nabla} |\phi(\vec{\lambda})\rangle = 0$

Parallel transport

$\boxed{\text{States in subspace } R(\vec{\lambda}) \text{ stay there !}}$

$\longrightarrow$ $\vec{A}_s(\vec{\lambda})$ is a (Berry) connection on a Fibre Bundle!

5- Fibre Bundle Language : Simple case of $U(1)$

* A G-bundle is a (E, π, M, F, G) also called a Principal bundle with $F \cong G$.

total space, projection onto M, base space, Fibre, Lie structure group

* Here $F = \{|\vec{\lambda}\rangle\}$ i.e vector space at $\vec{\lambda}$

* At each $\vec{\lambda} \in \mathcal{M}$, we have $U(1)$ invariance and thus a $U(1)$ bundle.

  equiv class

  Set of physical states $[|\vec{\lambda}\rangle] \equiv \{g|\vec{\lambda}\rangle, g \in U(1)\}$

* Projection $\pi(g|\vec{\lambda}\rangle) = \vec{\lambda}$

* Fix $g \longleftrightarrow$ choose a "section" $s: M \to E$ satisfying $\pi \circ s = id_M$

* Berry's connection is then

$$A = A_\mu d\lambda^\mu = \langle\vec{\lambda}|d|\vec{\lambda}\rangle \quad , \quad d = \frac{\partial}{\partial R^\mu} dR^\mu$$

* Berry's curvature, or field strength is

$$F = dA = \left(d\langle\vec{\lambda}|\right) \wedge \left(d|\vec{\lambda}\rangle\right)$$

$$= \left(\frac{\partial\langle\vec{\lambda}|}{\partial R^\mu}\right)\left(\frac{\partial|\vec{\lambda}\rangle}{\partial R^\nu}\right) dR^\mu \wedge dR^\nu$$

$\longrightarrow$ Gauge invariant and very important for TI's!

$\hookrightarrow$ $A \to A + df$ but $F \to F$

* In general, we will have $U(N)$ and a non-Abel connection $A \in U(N)$

6- Example : Spin in slowly varying $\vec{B}$

const mag

$$H(t) = \mu \vec{B}(t) \cdot \vec{\sigma}, \quad \vec{B}(t) = B_0 \hat{B}(t)$$

$\phi \in [0,\pi)$
$\theta \in [0,2\pi)$

* $\vec{\lambda} = \hat{B} = [\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi] \in S^2 \equiv \mathcal{M}$
* Adiabatic evolution : 2- bands that do not cross
* Subspace of lower band ; projection from

$$P(t) = \frac{1}{2}\left(\mathbb{1} - \hat{B}(t)\cdot\vec{\sigma}\right)$$

or

$$P(\vec{\lambda}) = \frac{1}{2}\left(\mathbb{1} - \vec{\lambda}\cdot\vec{\sigma}\right)$$

* Berry connection : $\quad A_s = [\vec{\nabla}P, P]\cdot\dot{\vec{\lambda}} = [\partial_t P, P]$

$$\mathcal{A}_s^{(i)} = [\partial_{\lambda_i} P, P] \qquad = \vec{\mathcal{A}}_s \cdot \dot{\vec{\lambda}}$$

$$= \left[-\frac{1}{2}\sigma_i, \frac{1}{2}\left(\mathbb{1} - \textstyle\sum_j \lambda_j \sigma_j\right)\right]$$

$$= \frac{1}{4}\sum_j \lambda_j [\sigma_i, \sigma_j]$$

$$= \frac{i}{2}\sum_{jk} \epsilon_{ijk}\lambda_j \sigma_k$$

* Plug this into $\dot{U}_A = \mathcal{A}_s U_A$ & solve :

In terms of t : $\quad \mathcal{A}_s = [\vec{\nabla}P, P]\cdot\dot{\vec{\lambda}} = [\partial_t P, P]$

$$\longrightarrow \quad \boxed{\dot{U}_A = \frac{i}{2}\sum_{ijk}\epsilon_{ijk}\dot{\lambda}_i \lambda_j \sigma_k U_A}$$

* We consider an evolution along the equator

$$\gamma : \{ (\theta(t), \phi(t)) = (\pi/2, 2\pi t) \mid t \in [0,1) \}$$

$$\rightarrow \lambda(t) = (\cos 2\pi t, \sin 2\pi t, 0)$$

$$\dot{\lambda}(t) = 2\pi(-\sin 2\pi t, \cos 2\pi t, 0)$$

$$\boxed{\begin{aligned} &\rightarrow \dot{U}_A = -i\pi \sigma_z U_A \\ &\rightarrow U_A(t) = e^{-i\pi t \sigma_z} \\ &\qquad = \cos \pi t \, \mathbb{1} - i \sin \pi t \, \sigma_z \end{aligned}}$$

* Initial state $|\psi(0)\rangle$, belonging to $P(0)$

$$|\psi(0)\rangle = |-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

* Evolved state is

$$|\psi(t)\rangle = U_A(t) |\psi(0)\rangle$$

$$= \cos \pi t \, |-\rangle - i \sin(\pi t) \, |+\rangle$$

* The state acquires a π-phase through a closed loop evolution

$$|\psi(0)\rangle = -|\psi(1)\rangle$$

7 - Using a Basis

* Now we have math understanding but we want something useful, i.e to do physics. So we need to assign a basis to our subspace

Let $$|\phi(\vec{\lambda})\rangle = \sum_n a_n(\vec{\lambda})\,|n(\vec{\lambda})\rangle$$

$$P(\lambda) = \sum_n |n(\lambda)\rangle\langle n(\vec{\lambda})|$$

Parallel transport implies (applying ⊘)

$$\vec{\nabla} a_m - i\sum_n A_{mn} a_n = 0$$

$$\vec{A}_{mn}(\vec{\lambda}) = i\langle m(\vec{\lambda})|\vec{\nabla}|n(\vec{\lambda})\rangle$$

Berry connection in a basis.

Solution to // transport : $a_m(\vec{\lambda}) = \sum_n \left[P_r \exp \int_\gamma d\vec{\lambda}\cdot\vec{A}\right]_{mn} a_n(0)$

$$= W_{mn}(\vec{\lambda})\, a_n(0)$$

Note : $$W_{mn}(\vec{\lambda}) = \langle m(\vec{\lambda})|\,U_a\,|n(0)\rangle$$

For a closed path, W is holonomy.

Next lecture, we discuss how to write useful expression for W and a related operator $\mathcal{W}$.
For closed γ, spectrum of W is gauge invariant

→ Under $U(N)$: $S|n\rangle$, $W = S^\dagger(\vec{\lambda})\, U_A(\vec{\lambda})\, S(\vec{0})$

So only when $\vec{\lambda}(t=0) = \vec{\lambda}(t=\infty)$

8- Relation between U_a and W

$$\mathcal{W}(\lambda) \equiv P(\lambda)\, U_A\, P(0) \quad : \quad // \text{ transp on } R = \bigcup_{\lambda} Im\left(P(\lambda)\right)$$

$\mathcal{W}(\lambda)$ and $W(\lambda)$ share same nonzero-part of spectrum

$\mathcal{W}(\lambda) \downarrow \in$ All Hilb

$W(\lambda) \downarrow$ subsp R

Now $\mathcal{W}(\lambda) = P(\lambda)\, U_a(\lambda)\, P(0) = U_a(\lambda)\, P(0)\, U_a^{\dagger}(\lambda)\, U_a(\lambda)\, P(0)$

$$= U_a(\lambda)\, P(0)$$

$$\Rightarrow \quad \nabla \mathcal{W} = [\nabla P, P]\, \mathcal{W}(\lambda) \quad ; \quad \mathcal{W}(0) = P(0)$$

- $\mathcal{W}(\lambda)$ and $U_a(\lambda)$ share same diff eq but different BC.

- $$\mathcal{W}(\lambda) \equiv \lim_{\Delta\lambda \to 0} P(\lambda)\, P(\lambda - \Delta\lambda)\, P(\lambda - 2\Delta\lambda) \cdots P(\Delta\lambda)\, P(0)$$
$$= \lim_{N \to \infty} \prod_{j=0}^{N} P\left(\lambda - \frac{j}{N}\lambda\right)$$
solves the D. E. w/ init cond $\mathcal{W}(0) = P(0)$

9- Example: Electric Polarization

Consider eigenstates of Bloch Ham $H(k)$ in 1D

$$H(k)\, u_{nk}(r) = E_n(k)\, u_{nk}(r)$$

$$H(k) = \frac{1}{2m}(p + k)^2 + V(r) \quad , \quad V(r+a) = V(a)$$

Full Eigenst :

$$|\psi_{nk}\rangle = \int dr\, \underbrace{e^{ikr}\, u_{nk}(r)}_{\psi_{nk}(r)}\, |r\rangle$$

Consider a small electric field $E = -\partial_t A = E_0$ $(A = -E_0 t)$. Minimal Coupling:

$$H(k(t)) = \frac{1}{2m}(p + k(t))^2 + V(r) \quad ; \quad k(t) = k + qE_0 t$$

Taking $\left|\frac{qE_0}{2m}\right| \ll \Delta$, we can apply adiabatic thm.

$$\rightarrow \quad u_{nk(t)} = W_{mn}(t)\, u_{mk(0)}$$

Consider average MB position

$$\langle \hat{\beta} \rangle = \langle \Psi_0 | e^{2\pi i \hat{X}/L} | \Psi_0 \rangle$$

where $|\Psi_0\rangle = \prod_{n,k} c^{\dagger}_{nk} |0\rangle$, and

$$c^{\dagger}_{nk} = \int_0^L dx\, \psi_{nk}(x)\, c^{\dagger}(x)$$

The position operator can be written as

$$\hat{X} = \int_0^L dx\, x\, c^{\dagger}(x) c(x)$$

$$\hat{\beta}\, c(x) = \exp\left[\int_0^L dy\, y\, c^{\dagger}(y) c(y)\right] c(x)$$

$$= e^{-2\pi i x/L}\, c(x)\, \hat{\beta}, \quad \text{so}$$

$$\hat{\beta}\, c_{nk}\, \hat{\beta}^{-1} = \int_0^L \psi^*_{nk}(x)\, e^{-2\pi i x/L}\, c(x)$$

$$\equiv \hat{c}_{nk}$$

So, $\langle \hat{\beta} \rangle = \langle \Psi_0 | \hat{\beta} \prod_{nk} c^\dagger_{nk} | 0 \rangle$

$$= \langle \Psi_0 | \prod_{nk} \tilde{c}^\dagger_{nk} | 0 \rangle$$

$$= \langle \Psi_0 | \tilde{\Psi}_0 \rangle$$

Writing it as a Slater determinant, we have

$$\langle \hat{\beta} \rangle = \det \left(\langle \Psi_{nk} | \tilde{\Psi}_{n'k'} \rangle \right)$$

$$= \det \left(\int_0^L dx \, u^*_{nk}(x) e^{-ikx} u_{n'k'}(x) e^{i(k' + \frac{2\pi}{L})x} \right)$$

$\longrightarrow$ $|\tilde{\Psi}_{n'k'}\rangle$ is Bloch wave with crystal momentum $k' + \frac{2\pi}{L}$ $\longrightarrow$ Conservation of cryst mom : $k' = k + \frac{2\pi}{L}$.

$$\langle \hat{\beta} \rangle = \det \left(\int_0^L dx \, u^*_{nk}(x) \, u_{n'(k - 2\pi/L)} \right)$$

In the lim $L \to \infty$, we have

$$\langle \beta \rangle = \det \left(W(2\pi) \right)$$

Gauge invarient Wilson loop in the BZ is related to the mean center of charge.

Further show that $P = \frac{q}{2\pi} \underbrace{\operatorname{Im} \log \det W}_{\text{Berry phase}}$

$\left(\langle \beta \rangle = e^{i \text{"} \langle x \rangle \text{"}} \right)$

Lecture 2 - A Topological Insulator

0- Recap last lecture

Adiabatic Transport & geometric phase

* Ingredients: $\mathcal{M} \in \lambda$, $H(\lambda)$, $V(\lambda) = \{ |\psi_i(\lambda)\rangle, \ i=1,\dots N\}$
  $S(\lambda) \subset V(\lambda)$, $P(\lambda): V(\lambda) \to S(\lambda)$

* Adiabatic evolution operator $U_a(t)\, P(0)\, U_a^\dagger(t)$
  $= P(\lambda(t))$

geometric $\quad U_a(t) = P\exp\left(\int_\gamma A_\mu \, d\lambda^\mu\right)$

* $A = A_\mu d\lambda^\mu$ is a conection on a $U(N)$ - bundle

* Gauge invariant curvature $F = dA + \underbrace{A \wedge A}_{\text{non-Abelian part}}$

* Instead of projectors, use states

$$\vec{A}_{\mu, mn} = \langle m | \, i \vec{\nabla} \, | n \rangle$$

* We also defined

$$W_{mn}(\vec{\lambda}) = \langle m(\lambda) | \, U_a \, | n(0) \rangle$$ (holonomy for closed path)

* Electric polarization is

$$P = \frac{q}{2\pi} \, \mathrm{Im} \log \det (W(2\pi))$$

Knowledge assumed :

* Tight-Binding
* Bloch Theorem

Revision :

Schrödinger Eq in $V(\vec{x}+\vec{R}) = V(\vec{x})$ yields solus $\Psi_{n\vec{k}}(\vec{r})$: n $\to$ band , $\vec{k} \in$ FBZ .

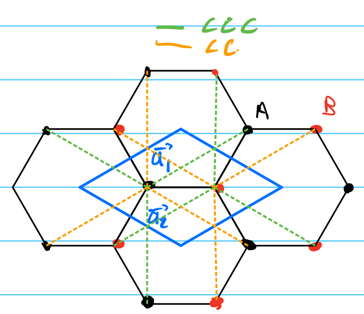
Bloch thm $\Psi_{n\vec{k}}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}}\, u_{n\vec{k}}(\vec{r})$, $u_{n\vec{k}}(\vec{r}+\vec{R}) = u_{n\vec{k}}(\vec{r})$

one can then work out pb $\langle \vec{k} | \hat{H} | \vec{k} \rangle \equiv H(\vec{k})$

$$H(\vec{k})\, u_{n\vec{k}}(\vec{r}) = E_n(\vec{k})\, u_{n\vec{k}}(\vec{r})$$

In the TB approx $u_{n\vec{k}}(\vec{r}) \longrightarrow |u_{n\vec{k}}\rangle = (u_{n1}, \dots u_{nN})^T$

3- The Haldane model (1988)

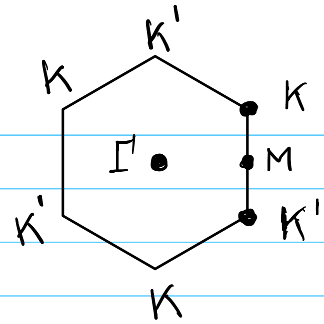


$$H = t \sum_{\langle ij \rangle} c_i^\dagger c_j + g \sum_{\langle\langle ij \rangle\rangle} e^{-i\nu_{ij}\phi} c_i^\dagger c_j + M \sum_i \epsilon_i c_i^\dagger c_i$$

$$\nu_{ij} = \begin{cases} +1 & \text{cc} \\ -1 & \text{ccc} \end{cases} , \quad \epsilon_i = \begin{cases} +1 & A \\ -1 & B \end{cases}$$

t : NN hopping
g : NNN hopping amplitude ,
ϕ : Haldane phase
M : Semenoff mass : staggered on-site energy

Brillouin zone :



$\longrightarrow$ $\mathcal{M} = \mathbb{T}^2$ compact manifold

$$H(\vec{k}) = \epsilon(\vec{k})\mathbb{1} + \vec{\sigma}\cdot\vec{d}(\vec{k}) \longrightarrow \begin{cases} \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \\ \vec{d} = (d_1, d_2, d_3) \end{cases}$$

Note : For the actual values of $\vec{d}$: HW (or check Bernevig's book)

two bands given by

$$E_\pm(\vec{k}) = \epsilon(\vec{k}) \pm d \quad , \qquad d = \|\vec{d}(\vec{k})\|$$

with eigenvectors

$$|\Psi_\pm(\vec{k})\rangle = \frac{1}{\sqrt{2d(d\pm d_3)}}\begin{pmatrix} d_3 \pm d \\ d_1 + i d_2 \end{pmatrix}$$

However, it's not unique! Each eigenstate has a $U(1)$ symmetry. E.g. for $|\Psi_-\rangle$:

$$|\tilde{\Psi}_-\rangle = \frac{-d_1 + i d_2}{\sqrt{d_1^2 + d_2^2}}|\Psi_-\rangle = \frac{-1}{\sqrt{2d(d+d_3)}}\begin{pmatrix} d_1 - i d_2 \\ d_3 + d \end{pmatrix}$$

$$\longrightarrow \boxed{|\tilde{\Psi}_-\rangle = e^{i\phi}|\Psi_-\rangle} \qquad U(1) \text{ invariance}$$

We are interested in the insulating phase, so the lower band is completely filled and forms our occupied subspace $S(\vec{k})$:

$$P_-(\vec{k}) = |\Psi_-(\vec{k})\rangle\langle\Psi_-(\vec{k})|$$

4- "Berryology" of the Chern Insulator

The $U(1)$ Berry connection is

$\vec{A}_-(\vec{k}) = i \langle \psi_-(\vec{k}) | \partial_{\vec{k}} | \psi_-(\vec{k}) \rangle$, which transforms as

$\tilde{\vec{A}}_-(\vec{k}) = i \langle \psi_-(\vec{k}) | \partial_{\vec{k}} | \psi_-(\vec{k}) \rangle - \partial_{\vec{k}} \phi$ under $U(1)$

$$\vec{A}_-(\vec{k}) = \frac{d_1 \vec{\nabla} d_2 - d_2 \vec{\nabla} d_1}{2\, d\, (d - d_3)}$$ HW

The $U(1)$ (gauge invariant) Berry curvature is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\boxed{F_{\mu\nu} = \frac{1}{2d^3} \epsilon^{ijk} d_i \partial_\mu d_j \partial_\nu d_k}$$

Also HW

$[\vec{d} \cdot (\partial_\mu \vec{d} \times \partial_\nu \vec{d})]$

Interpreations in terms of topology :

$\boxed{A}$ Consider the map $f : T^2 \to S^2$ given by

$$\vec{f}(\vec{k}) = \frac{\vec{d}}{d}$$

The Jacobian of this map is $2\times$ the Berry curvature

$\longrightarrow \quad \int F = \frac{1}{2} \text{Area} \times C_1 = 2\pi C_1$

C_1 : counts nr of times f winds around origin

$\boxed{B}$ Algebraic Topology & Differential Geometry Picture

The name "Chern" comes from the fact that F is associated to the first Chern class. which is the 2^{nd} cohomology class over integers:

$$H^2(\mathbb{T}^2, \mathbb{Z}).$$

This bit means $F = dA$ is only true locally

It's a topological invariant associated with the curvature of the connection on the $U(1)$ bundle.

$\boxed{C}$ Implication: presence of a "magnetic monopole"

$$F = [\vec{\nabla} \times \vec{A}] \cdot d\vec{S} \equiv \vec{B} \cdot d\vec{S}$$

"Gauss" law for $\vec{B}$: $\oint_{\mathcal{M}} \vec{B} \cdot d\vec{S} = Q_B$

Nonzero F integral implies the presence of a monopole

$$\oint_{\mathcal{M}} \vec{B} \cdot d\vec{S} = \int_{\Sigma} dV\, \vec{\nabla} \cdot \vec{B} = \int dV\, \rho_m \quad \Big| \quad \mathcal{M} = \partial\Sigma$$

Monopole of charge g inside sphere creates nonzero divergence

$$\boxed{\vec{\nabla} \cdot \vec{B} = g\,\delta(\vec{x})}$$

5 - Physical Implications: Quantized Hall conductance

We will show how to relate the Hall conductance of a 2D system to the integral of the Berry curvature:

$$\sigma_{xy} = \frac{e^2}{h} \frac{1}{2\pi} \int_{BZ} dk_x\, dk_y\, F_{xy}(k_x, k_y)$$

Given a Hamiltonian with translation sym.

$$H_0 = \sum_{ij\alpha\beta} c^\dagger_{i\alpha} h^{\alpha\beta}_{ij} c_{j\beta}$$

i,j : spatial indices
α,β : internal indices

$$= \sum_{k\alpha\beta} c^\dagger_{k\alpha} h^{\alpha\beta}_{k} c_{k\beta}$$

It couples to an external field through Peierls substitution

$$h^{\alpha\beta}_{ij} \longrightarrow h^{\alpha\beta}_{ij}\, e^{i\int_{x_i}^{x_j} d\vec{l}\cdot\vec{A}(\vec{l})}$$

Under a set of reasonable approximations (see Bernevig), this gives

$$H = H_0 + \sum_{kq\alpha\beta} c^\dagger_{k+q/2} \frac{\partial h^{\alpha\beta}}{\partial \vec{k}}\cdot\vec{A}_{-q}\, c_{k-q/2}$$

The current operator couples to the external field.

$$\vec{j}_q = \sum_{k\alpha\beta} c^\dagger_{k+q/2} \frac{\partial h^{\alpha\beta}}{\partial \vec{k}} c_{k-q/2}$$

i.e

$$H = H_0 + \sum_{\vec{q}} \vec{j}_q \cdot \vec{A}_{-q}$$

From Linear Response Theory, it can be shown that the linear response of a MB system in the GS is

$$\langle j_i(x,t)\rangle_0 = \langle j_i(x)\rangle_0 + \int_{-\infty}^{\infty} dt'\, d^dx' \sum_{ij} R_{ij}(x-x', t-t')\, A_j(x',t')$$

$$R_{ij}(x-x',t-t') = -i\,\theta(t-t')\langle 0|[j_i(x,t), j_j(x',t')]|0\rangle + \text{diag part.}$$

we don't care about this for an insulator.

We want to find σ_{ij} in Ohm's law:

$$j_i(q,\omega) = i\omega\,\sigma_{ij}(q,\omega)\, A_j(k,\omega)$$
$$= R_{ij}(k,\omega)\, A_j(k,\omega)$$

$(A = E/i\omega)$

Next steps are lengthy, but require

* Fluctuation - Dissipation theorem
  → Relates response kernel to the retarded Green's Function (GF)

* Computing imaginary-time GF (easier) & then perform analytic-continuation $\omega_m \to \omega + i\eta$

$$R_{ij}(q, \nu_m) = -\frac{1}{\beta^2} \sum_{k, n} \mathrm{Tr}\left(\frac{\partial h}{\partial k_i} G^T(k - q/2, i\omega_n) \frac{\partial h}{\partial k_j} G^T(k - q/2, i\omega_{n-m}) \right)$$

$$\sigma_{ij}(q, \omega) = \frac{R_{ij}(q, \nu_m \to \omega + i\eta)}{i\omega}$$

* Perform an adiabatic transformation to a flat-band basis with $E_\pm = \pm 1$, which leads to

Gauge invariant

$$\longrightarrow \sigma_H \equiv \sigma_{ij}(\omega \to 0) = \frac{1}{V} \sum_{k} \mathrm{Tr}\left[(\partial_i P_G)(\partial_j P_G) P_G \right]$$

where P_G is the projector onto the occupied band.

* When written in terms of Bloch states,

Requires smooth gauge for $|\psi_k\rangle$

$$\boxed{\begin{aligned} \sigma_H &= \int \frac{d^2k}{(2\pi)^2} \, -i\left[\langle \partial_i \psi_k | \partial_j \psi_k \rangle - \langle \partial_j \psi_k | \partial_i \psi_k \rangle \right] \\ &= \frac{1}{2\pi} \int d^2k \, F_{xy} \qquad (\text{in units of } e^2/h) \end{aligned}}$$

We have established that $\sigma_H = \frac{e^2}{h} C_1$

First Chern #

This (in the case of many bands for Landau-levels), is the famous TKNN formula.

6- Chern - Simons Action

The Hall conductivity appears as

$$j_i = \sigma_H \varepsilon_{ij} E_j$$

The continuity equation dictates the following

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{j} = -\sigma_H \left(\partial_x E_y - \partial_y E_x \right)$$

$$\begin{vmatrix} \hat{z} \\ \partial_x & \partial_y \\ E_x & E_y \end{vmatrix}$$

$\downarrow$ Maxwell eq $\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E}$

$$\frac{\partial \rho}{\partial t} = \sigma_H \frac{\partial B_z}{\partial t}$$

Covariant Formulation : $j^\mu = \sigma_H \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda$

Consider the Chern-Simons action

$$S = \frac{\sigma_H}{2} \int dt \int d^2x \, A_\mu \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda$$

of a background gauge field. This gives precisely

$$j^\mu = \frac{\delta S}{\delta A_\mu}$$

At the quantum level, This forces σ_H to be quantized because a gauge transformation $A_\mu \to A_\mu + \partial_\mu f$ means

$$\frac{2}{\sigma_H} \mathcal{L}' = \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + \epsilon^{\mu\nu\lambda} \partial_\mu f \partial_\nu A_\lambda$$

$$S' = S + \boxed{\frac{\sigma_H}{2} \epsilon^{\mu\nu\lambda} \int d^2x \, dt \, \partial_\mu \left[f \, \partial_\nu A_\lambda \right]}$$

some gauge choice $\sim 2\pi k$

$e^{iS} \to e^{iS}$ ✓

* When there is obstruction, total deriv not nec 0, but a careful analysis using Euclidean FT leads to quant cond
* Integrating out Fermions of Chern insulator coupled to A in linear resp leads to CS.

Lecture 3 - Topological Insulator (continued)

0- Recap last lectures

(1) Adiabatic Transport & geometric phase

* Ingredients : $\mathcal{M} \in \lambda$, $H(\lambda)$, $V(\lambda) = \{ |\psi_i(\lambda)\rangle , i=1,\dots N \}$
$S(\lambda) \subset V(\lambda)$, $P(\lambda) : V(\lambda) \to S(\lambda)$

* Adiabatic evolution operator $\mathcal{U}_a(t)\, P(0)\, \mathcal{U}_a^\dagger(t)$
$= P(\lambda(t))$

geometric $\mathcal{U}_a(t) = P \exp\left(\int_\gamma A_\mu \, d\lambda^\mu \right)$

* $A = A_\mu d\lambda^\mu$ is a conection on a $\mathcal{U}(N)$ - bundle

* Gauge invariant curvature $F = dA + A \wedge A$

(2) Haldane model for Chern Insulators

$$H = \vec{d}(\vec{k}) \cdot \vec{\sigma}$$

* Occupied subspace $P_-(\vec{k}) = |\psi_-(\vec{k})\rangle\langle\psi_-(\vec{k})|$,

* $\mathcal{U}(1)$ symmetry $|\psi_-\rangle \to e^{i\phi(\vec{k})} |\psi_-(\vec{k})\rangle$, $\phi = \arctan(-d_2/d_1)$
$\left(e^{i\phi(\vec{k})} = -\hat{d}_1 + i\hat{d}_2 \right)$

* Berry Connection :

$$\vec{A}_-(\vec{k}) = \frac{d_1 \vec{\nabla} d_2 - d_2 \vec{\nabla} d_1}{2\, d\,(d - d_3)}$$

* Berry Curvature :

$$F_{\mu\nu} = \frac{1}{2d^3} \epsilon^{ijk} d_i \partial_\mu d_j \partial_\nu d_k$$

* First Chern number : integral of $F_{\mu\nu}$

$$C_1 = \frac{1}{2\pi} \int_{T^2} F$$

A) C_1 counts winding of map $f : T^2 \to S^2$
$\vec{k} \mapsto \vec{d}(\vec{k}) / \| \vec{d}(\vec{k}) \|$

B) F is the first Chern class $\to$ 2^{nd} cohomology class over integers $H^2(T^2, \mathbb{Z})$

C) Nonzero F necessarily implies "monopoles"

* Quantized Hall conductance through TKNN formula

$$\sigma_{xy} = \frac{e^2}{h} C_1$$

* σ_{xy} is quantized to avoid quantum anomaly of Chern-Simons action.

$$S = \frac{\sigma_H}{2} \int d^2x \, dt \, \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda$$

1- Obstruction to Wannierization

BZ has no boundary, so Stoke's theorem fails to work when integrating over all the BZ

$$\boxed{\int_{BZ} F \neq \int_{\partial BZ} A \quad \text{if} \quad \int_{BZ} F \neq 0}$$

(For $\mathbb{Z}_2$ insulators, obstruction is in half the BZ)

This means $\boxed{C_1 \neq 0 \Rightarrow \vec{A}(\vec{k}) \text{ not smooth}}$

Consider BZ split into $\underset{\uparrow}{R}$ & $\bar{R}$

A_1 smooth (on R), A_2 smooth (on $\bar{R}$)

At the boundary between R & $\bar{R}$,

$$|\psi_1\rangle = e^{i(\theta-\phi)}|\psi_2\rangle = e^{i\chi}|\psi_2\rangle$$

where $|\psi_1\rangle = e^{i\phi}\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}$, $|\psi_2\rangle = e^{i\theta}\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}$

The Berry potential is

$$\vec{A}_2(\vec{k}) = \vec{A}_1(\vec{k}) + \nabla\chi(\vec{k})$$

So, writing

$$\sigma_{xy} = \frac{e^2}{h}\frac{1}{2\pi}\left(\int_R F + \int_{\bar{R}} F\right)$$

$$= \frac{e^2}{h}\frac{1}{2\pi}\left(\int_{\partial R} A_1 + \int_{\partial\bar{R}} A_2\right)$$

$\partial R = -\partial\bar{R}$

$$= \frac{e^2}{h} \frac{1}{2\pi} \underbrace{\left(\int_{\partial R} d\vec{k} \cdot \vec{\nabla} \chi \right)}_{\substack{\text{winding number} \\ \text{of gauge trans on} \\ \text{boundary of } R}} = \frac{e^2}{h} n$$

$\longrightarrow$ $n \in \mathbb{Z}$ because of single-valuedness constraint of wavefunction.

If we look at our 2-band model and

$$\vec{f}(\vec{k}) = \frac{\vec{d}}{\|\vec{d}\|} \equiv f(\theta, \phi)$$

when $\hat{d}_3 = \cos\theta > 0$, $C_1 = 0$. This gauge choice is smooth:

$$|\psi_1\rangle = \frac{1}{\sqrt{2(1-\hat{d}_3)}} \begin{pmatrix} \hat{d}_3 - 1 \\ \hat{d}_1 + i\hat{d}_2 \end{pmatrix}$$

When $\cos\theta \in [-1, 0]$. $C_1 \neq 0$. We have a problem: at $\theta = 0$:

$$\lim_{\theta \to 0} |\psi_1\rangle = \begin{pmatrix} -\theta/2 \\ e^{-i\phi} \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 \\ +1 \end{pmatrix} & \phi = 0 \\ \begin{pmatrix} 0 \\ -1 \end{pmatrix} & \phi = \pi \end{cases}$$

But they represent same point in space $\rightarrow$ multivalued.

The other gauge choice

$$|\psi_2\rangle = \frac{1}{\sqrt{2(1+\hat{d}_3)}} \begin{pmatrix} \hat{d}_1 - i\hat{d}_2 \\ -1 - \hat{d}_3 \end{pmatrix}$$

has the same problem but at $\theta = \pi$.

You can now see why we cannot define Wannier functions:

$$|w_n(\vec{r})\rangle = A_{cell} \int_{BZ} \frac{d\vec{k}}{(2\pi)^2} \, |\Psi_n(\vec{k})\rangle$$

3 – Determining TPT from Dirac Hamiltonian

Generic Dirac Hamiltonian can be written as

$$h(\vec{k}) = \sum_{i,j=1}^{2} k_i T_{ij} \sigma_j \quad + \underbrace{M\sigma_3}_{}$$

Dirac gap / mass term

$$d_1 = k_x T_{11} + k_y T_{21}$$
$$d_2 = k_x T_{12} + k_y T_{22} \qquad , \qquad d_3 = M$$

$$F_{xy} = \frac{1}{2d^3} \epsilon^{ijk} d_i \partial_x d_j \partial_y d_k$$
$$= \frac{1}{2d^3} M [T_{11} T_{22} - T_{21} T_{12}] = \frac{1}{2d^3} M \det(T)$$

note $d^3 = \left[(k_x T_{11} + k_y T_{21})^2 + (k_x T_{12} + k_y T_{22})^2 + M\right]^{3/2}$

By using the identities

$$\frac{1}{2}\int_{-\infty}^{\infty} dx \, \frac{m}{(m^2 + x^2)^{3/2}} = \text{sign}(m)$$

$$\int_0^{2\pi} \frac{d\theta}{\sum_j T_{1j}^2 \cos^2\theta + \sum_j T_{2j}^2 \sin^2\theta + \sum_j T_{1j} T_{2j} \sin(2\theta)} = \frac{2\pi}{|\det(T)|}$$

You can show that

$$\boxed{\sigma_{xy} = \frac{1}{2}\,\mathrm{sign}(M)\,\mathrm{sign}(\det(T))}$$

Note :

- σ_{xy} half-quantized because continuum is blind to the lattice, which is a regulator.
- $\int_{\mathcal{M}} F = 2\pi C$ only if $\mathcal{M}$ is a compact manifold,
- The "high-energy" fermions from bending of bands contribute to the other half σ_{xy} : "spectator fermions"

For the Haldane model, gap closing (indicating a transition) happen at points K & K'. Near those points:

$$H(\vec{K_i} + \vec{k}) = -3t_2\cos\phi \pm \frac{3}{2}t_1\left(k_y\sigma_x \mp k_x\sigma_y\right) + \underbrace{\left(M - 3\sqrt{3}\,t_2\sin\phi\right)\sigma_z}_{\text{gapping term}}$$

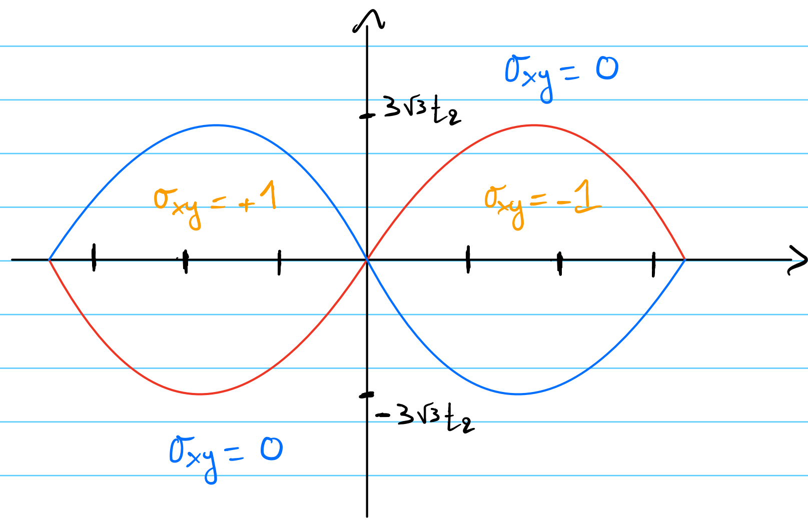
* At $|M| \to \infty$, trivial insulator : fully localized on sites A or B : $\sigma_{xy} = 0$

* At $M = 3\sqrt{3}\,t_2\sin\phi$, the gap closes and we go

from $\sigma_{xy} = \frac{1}{2}\,\mathrm{sign}(M - 3\sqrt{3}\,t_2\sin\phi) = \mp\frac{1}{2}$ to $\pm\frac{1}{2}$ $\left(\begin{matrix}\phi<0\\ \phi>0\end{matrix}\right)$

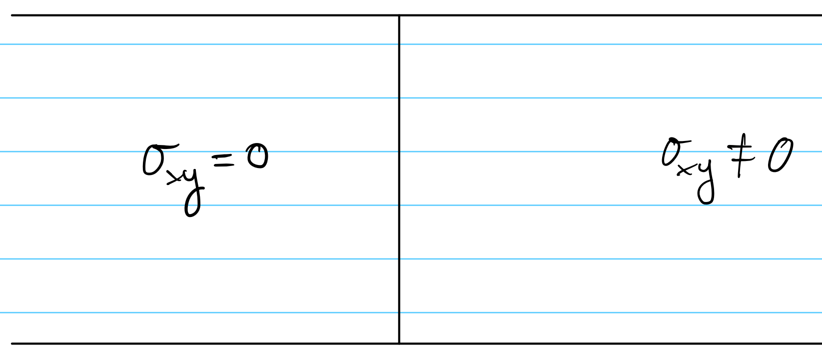
$\to \Delta\sigma_{xy} = \pm 1$ which means now $\sigma_{xy} = \pm 1$ (on lattice)

So we have the following phase diagram



4 - Gapless chiral edge modes

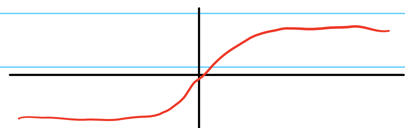
Consider two infinite samples with their interface along the y-direction :



Since we know that the mass term is responsible for a TPT, we will model it with some function

$$m(x) \begin{cases} > 0 & x > 0 \\ < 0 & x < 0 \end{cases} \quad \text{bounded at } \pm\infty$$

e.g :



$$H(x) = -i\partial_x \sigma_x - i\partial_y \sigma_y + m(x)\sigma_z$$

Ansatz:
$$\Psi(x,y) = \phi_1(x)\,\phi_2(y)$$
$$= \phi_2(y)\, e^{-\int_0^x m(x)\,dx'}$$

$\phi_2(y)$ must be a 2-component spinor:

$$E\phi_2(y) = -i\sigma_y \partial_y \phi_2(y) + m(x)\left(i\sigma_x \phi_2(y) + \sigma_z \phi_2(y)\right)$$

* At $E=0$, we must impose

$$i\sigma_x \phi_2 = -\sigma_z \phi_2$$

$$\Rightarrow \quad \phi_2(y) = \chi(y)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$$

* Then, $E\chi(y) = -i\partial_y \chi(y)$ is solved by $\chi(y) = e^{ik_y y}$, with $E = k_y$

$$\Longrightarrow \quad \Psi(x,y) = e^{ik_y y}\, e^{-\int_0^x m(x)\,dx'}\, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$$

1) Localization at domain wall $x=0$

2) Propagation in $+y$ direction $\rightarrow$ chiral

2 - The role of symmetry

Starting from graphene semimetal, we will see that by breaking its symmetries, we open different types of gaps

$$H(\vec{k}) = \begin{pmatrix} 0 & -t\left(1 + e^{i\vec{k}\cdot\vec{a}_1} + e^{i\vec{k}\cdot\vec{a}_2}\right) \\ -t\left(1 + e^{-i\vec{k}\cdot\vec{a}_1} + e^{-i\vec{k}\cdot\vec{a}_2}\right) & 0 \end{pmatrix}$$

$\longrightarrow$ two Dirac cones at $K_\pm = \frac{2\pi}{3a}\left(1, \pm\frac{1}{\sqrt{3}}\right)$

$$H(\vec{K}_\pm + \vec{k}) = k_x \sigma_x + k_y \sigma_y$$

Note : $\vec{K}$ and $\vec{K}'$ $(=K_-)$ are TR partners

(A) Inversion - Symmetry

$$I : (x, y) \longrightarrow (-x, -y)$$

$$\left.\begin{aligned} I c_{iA} I^{-1} &= c_{-i,B} \\ I c_{iB} I^{-1} &= c_{-i,A} \end{aligned}\right\} \quad I c_{i\alpha} I^{-1} = \sigma^x_{\alpha\alpha'} c_{-i\alpha'}$$

$$I H I = H, \qquad H = \sum_k c_k H(k) c_k$$

means

$$\sigma_x H(\vec{k}) \sigma_x = H(-\vec{k})$$

(B) Time-Reversal-Symmetry

Spinless fermions : $T H(\vec{k}) T = H(-\vec{k})$
$H^*(\vec{k}) = H(-\vec{k})$

fixed by TRS
$$\begin{cases} H^*(\vec{K}+\vec{k}) = k_x \sigma_x - k_y \sigma_y = H(-\vec{K}-\vec{k}) \\ H(\vec{K}'+k) = -k_x \sigma_x + k_y \sigma_y \end{cases}$$

Note : Dirac cone not protected by TRS or inversion separately :

Inversion breaking (but TRS sym) : $m\sigma_z \rightarrow$ opens gap of size $2m$.

TRS breaking (but inv sym) : $\tilde{m}\sigma_z$ term such that

$$\left.\begin{aligned} H(\vec{K}+k) &= k_x \sigma_x + k_y \sigma_y + \tilde{m}\sigma_z \\ H(\vec{K}'+\vec{k}) &= -k_x \sigma_x + k_y \sigma_z - \tilde{m}\sigma_z \end{aligned}\right\} \text{" Haldane mass "}$$

Stability of Dirac Nodes

However, with both IT : no σ_z term allowed and Dirac cones are protected. IT provides local stability of Dirac cones.

* In that case, K & K' carry a vorticity $\oint dk A = \pm\pi$ which is the Berry phase

* If perturbations are large enough, Dirac cones can annihilate each other.

* C_3 sym (actually ITC_3) forces Dirac points to be fixed $\longrightarrow$ Global Stability