Field Theory in Condensed Matter (2023-2024)

Exercise Series on Quantum Geometry and Topology – 2

1 "Magnetic monopole"

Consider again the simple example of a spin 1/2 coupled to an external magnetic field,

$$H = -\mu \boldsymbol{\sigma} \cdot \mathbf{B},$$

where

$$\boldsymbol{B} = B\left[\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta\right].$$

a) Show that the lower band's eigenstate can be written as

$$|\psi_-\rangle = \begin{pmatrix} e^{\phi} \sin\theta/2 \\ -\cos\theta/2 \end{pmatrix}$$

b) Calculate the Berry connection $\mathbf{A} = i \langle \psi_- | \nabla | \psi_- \rangle$

c) Show that the Berry curvature, when expressed in terms of the coordinates $B = (B_x, B_y, B_z)$ is given by

$$F_{ij} = -\frac{1}{2}\epsilon_{ijk}\frac{B_k}{B^3}$$

This is precisely the field of a point particle (monopole) in the space of magnetic fields, sitting at $\mathbf{B} = 0$ and of strength $q = -1/2$.
Note: it is slightly confusing to say "this is a magnetic monopole in the space of magnetic fields"!

2 Handy expressions

Consider a generic 2-band model in 2D, described by the Hamiltonian

$$H(\mathbf{k}) = \epsilon(\mathbf{k}) + \mathbf{d}(\mathbf{k}) \cdot \boldsymbol{\sigma}.$$

a) Show that the Berry connection of the lower band, in the gauge we first derived in class and in the previous exercise, is given by

$$A_\mu = \frac{d_2 \partial \mu d_1 - d_1 \partial_\mu d_2}{2d(d - d_3)}, \tag{1}$$

where $\mu = 1, 2$ denotes the x and y directions in the Brillouin zone.

b) Show that the Berry curvature is given by

$$\begin{aligned} F_{\mu\nu} &= \frac{1}{2d^3}\epsilon^{ijk} d_i \partial_\mu d_j \partial_\nu d_k \\ &= \frac{1}{2}\hat{\mathbf{d}} \cdot \left[\partial_\mu \hat{\mathbf{d}} \times \partial_\nu \hat{\mathbf{d}}\right] \end{aligned}$$

c) Consider a generic continuum Dirac Hamiltonian described by

$$H(\mathbf{k}) = \sum_{i,j=1}^{2} k_i T_{ij} \sigma_j + M\sigma_3$$

where T is a 2×2 matrix. Show that an approximation to the Chern number using this low-energy Dirac Hamiltonian leads to

$$C \sim \frac{1}{2\pi}\int_{\mathbb{R}^2} F_{12} d^2k = \frac{1}{2}\mathrm{sign}(M)\mathrm{sign}(\det T). \tag{2}$$

The reason for the half-quantization is that we are omitting the bending of the band present in the lattice. These so-called spectator fermions are the reason we are missing another half to obtain an integer. This approximation is still useful as any changes in Chern number driven by a topological phase transition is still integer-valued.

3 Determining topological phase transitions of the Haldane model

Consider the Hamiltonian for the Haldane model of a Chern insulator on a honeycomb lattice,

$$H = t\sum_{\langle ij\rangle} c_i^\dagger c_j + g\sum_{\langle\langle ij\rangle\rangle} e^{i\nu_{ij}\phi} c_i^\dagger c_j + M\sum_i \epsilon_i c_i^\dagger c_i, \tag{3}$$

where t is the real nearest neighbor hopping, g is the amplitude of the complex next-nearest hopping, with its phase determined by whether $\mathbf{x}_i$ and $\mathbf{x}_j$ are connected clockwise, or counter clockwise, e.g. $\nu_{ij} = 1$ for a clockwise connection and $\nu_{ij} = -1$ otherwise. The last term is called the Semenoff mass term, a staggered on-site potential, with $\epsilon_i = 1$ on A-sites, and $\epsilon_i = -1$ on B-sites.

a) Show that the coefficients of the Bloch Hamiltonian $H(\mathbf{k}) = \epsilon(\mathbf{k}) + \mathbf{d}(\mathbf{k}) \cdot \boldsymbol{\sigma}$, where the 2×2 matrix structure comes from the (A, B) sublattices, are given by

$$\begin{aligned}
\epsilon(\mathbf{k}) &= 2g\cos(\phi)\sum_{i=1}^{3}\cos(\mathbf{k}\cdot\mathbf{a}_i), \\
d_1(\mathbf{k}) &= t + t\sum_{i=1}^{2}\cos(\mathbf{k}\cdot\mathbf{a}_i), \\
d_2(\mathbf{k}) &= t\sum_{i=1}^{2}\sin(\mathbf{k}\cdot\mathbf{a}_i), \\
d_3(\mathbf{k}) &= M + 2g\sin(\phi)\left[\sin(\mathbf{k}\cdot\mathbf{a}_1) - \sin(\mathbf{k}\cdot\mathbf{a}_2) - \sin(\mathbf{k}\cdot\mathbf{a}_3)\right]
\end{aligned} \tag{4}$$

where $\mathbf{a}_1$ and $\mathbf{a}_2$ are the hexagonal lattice vectors and $\mathbf{a}_3 = \mathbf{a}_1 - \mathbf{a}_2$.

b) Show that for $M = g = 0$, the Hamiltonian has time-reversal symmetry, inversion symmetry, and C_3 symmetry, which is a discrete rotation by 120°. Show that the last symmetry implies that the system is gapless at the $\mathbf{K} = 2\pi/3a(1, 1/\sqrt{3})$ point in the Brillouin zone and at its time-reversal partner point $\mathbf{K}'$.

c) Verify that the g term breaks time-reversal symmetry and that the M term breaks inversion symmetry. The only symmetry left in this system is a discrete rotation by 120°, C_3. Thus the gap-closing transitions must happen at the $\mathbf{K}$ and $\mathbf{K}'$ points.

d) By expanding the Hamiltonian to linear order in $\mathbf{k}$ around the gap-closing points, derive the following two continuum Dirac Hamlitonians

$$\begin{aligned}
H(\mathbf{K}+\mathbf{k}) &= -3g\cos(\phi) - \frac{3}{2}t\,(k_x\sigma_y - k_y\sigma_x) + \left[M - 3\sqrt{3}g\sin(\phi)\right]\sigma_z, \\
H(\mathbf{K}'+\mathbf{k}) &= -3g\cos(\phi) - \frac{3}{2}t\,(k_x\sigma_y + k_y\sigma_x) + \left[M + 3\sqrt{3}g\sin(\phi)\right]\sigma_z.
\end{aligned}$$

Where do the topological phase transitions take place?

e) By using Eq. (2), argue about which side of the transitions you found above corresponds to a topologically nontrivial phase and which side corresponds to a topologically trivial phase.