Field Theory in Condensed Matter (2023-2024)

Exercise Series on Quantum Geometry and Topology

1 Geometry of Quantum Mechanics

In this exercise we prove some of the results that were discussed during the lecture on the Berry phase and the geometry of quantum mechanics. Consider a generic, adiabatically evolving Hamiltonian $H(\boldsymbol{\lambda})$, with a set of time-dependent parameters $\boldsymbol{\lambda}(t) \in \mathcal{M}$, where $\mathcal{M}$ is a smooth manifold. Let $P(t)$ be the projector onto the subspace of instantaneous eigenstates $R = \{|\psi_n(t)\rangle\,; n = 1, \cdots, M\}$, that lie below a certain energy E, i.e. $\{E_n(t); |E_n(t) - E| < \Delta,\ \forall t \in [0,1]\}$, for some $\Delta > 0$. In the adiabatic limit, we have

$$P(t) = U_a(t)P(0)U_a^\dagger(t),$$

where $U_a(t)$ is the adiabatic evolution operator.

a) Show that

$$\dot{P} = \left[\dot{U}_a U_a^\dagger, P\right] \tag{1}$$

$P\dot{P}P = 0$. The rest follows.

b) Show that Eq. (1) is satisfied if $\dot{U}_a U_a^\dagger = \left[\dot{P}, P\right]$ and that the adiabatic evolution operator is given by

$$\begin{aligned} U_a(t) &= \mathcal{T}\exp\left[\int_0^t \mathcal{A}(t')dt'\right] \\ &= \mathcal{P}\exp\left[\int_\gamma \mathbf{A}\cdot d\boldsymbol{\lambda}\right], \end{aligned}$$

where γ is a path through $\mathcal{M}$ parametrized by t. Give an expression for the Berry connection $\mathbf{A}$.

c) Let $|\phi(\boldsymbol{\lambda})\rangle \in R$. Express $\nabla|\phi(\boldsymbol{\lambda})\rangle$ in terms of P and $\dot{P}$.

d) Derive the parallel transport condition

$$[\nabla - \mathbf{A}]\,|\phi(\boldsymbol{\lambda}\rangle = 0$$

and show that $P\nabla|\phi\rangle = 0$. Interpret the results.

2 Spin in a Magnetic Field

Consider a time-dependent Hamiltonian $H(t)$ of a spin degree of freedom $-\mu\boldsymbol{\sigma}$ coupled to a varying magnetic field $\mathbf{B}(t)$,

$$H(t) = -\mu\boldsymbol{\sigma} \cdot \mathbf{B}(t),$$

where the magnetic field is constant in magnitude but can change directions, $\mathbf{B} = B_0\hat{\boldsymbol{\lambda}}(t)$, where

$$\hat{\boldsymbol{\lambda}}(t) = B_0 \left[\cos\phi(t)\sin\theta(t), \sin\phi(t)\sin\theta(t), \cos\theta(t)\right].$$

a) Show that the projector onto the lower energy eigenspace can be written as

$$P(t) = \frac{1}{2}\left(1 - \hat{\boldsymbol{\lambda}}(t)\cdot\boldsymbol{\sigma}\right)$$

b) Show that the Berry connection is given by

$$A_i = \frac{i}{2}\epsilon_{ijk}\lambda_j\sigma_k,$$

where repeated indices are summed over.

c) Consider now a system constrained to move along the equator. Show that

$$U_a(t) = \exp\{-i\pi t\sigma_z\}$$

d) Finally, show that this leads to eigenstates acquiring a π phase under a closed-loop evolution.

e) Instead of projectors, let's work with a basis. Show that the eigenstate of the lower band is given by

$$|\psi_-(t)\rangle = \frac{1}{\sqrt{2(1-\cos\theta(t))}}\begin{pmatrix}\cos\theta(t) - 1\\ e^{\phi(t)}\sin\theta(t)\end{pmatrix}$$

f) Calculate the Berry connection $\mathbf{A} = i\langle\psi_-|\nabla|\psi_-\rangle$ and show that indeed

$$\oint_\gamma \mathbf{A}\cdot d\boldsymbol{\lambda} = \pi$$

where γ is the equator.

e) Determine the Berry curvature $\mathbf{F}$ by using Stokes' theorem.
Note: this term usually appears as an "anomalous" contribution to the equation of motion of a spin, or to semiclassical equations of motion for electrons moving in a periodic potential.